Lesson 2.x – Trigonometric Functions (basics)

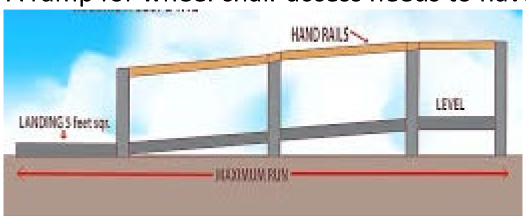
When we looked at geometric shapes, we measured the size of angles by degrees – 90 degrees is a right angle, for example. For comparison, a 5 degree angle is 'small'. Perhaps you have wondered how the lengths of the sides for a degree relate to the number of degrees. Trigonometry will help us find answers.

We will start by looking at two similar right triangles

Reminders: 360 degrees is a complete 'revolution'.
Ramps need to have a small slope for safety.

Part A:

A ramp for wheel chair access needs to have an angle of 5 degrees.

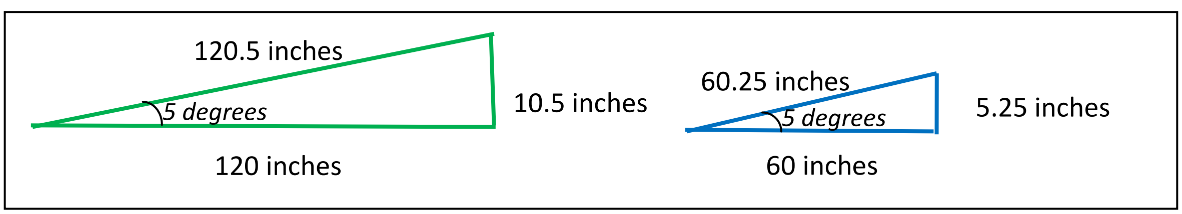


Here is the sloped portion of the ramp

Here is the ramp to the first support:

Both of these portions of the ramp are right triangles. We might think we need to know two of the three sides in order to find the third (the Pythagorean Theorem: $a^2 + b^2 = c^2$). However, if we know the angle size then we only need to know one of the three lengths in order to find the other two. Ratios will be the key … ratios that are "trigonometric functions".

Here are the two triangles from the images of the ramp (entire ramp; first half). Note that these triangles are not drawn to scale; this helps us see the height of the smaller triangle and the angle at the start of the ramp in both triangles.



Questions:

1. For the larger triangle, write the ratio of the height to the base (bottom side). Calculate the decimal for this ratio to 4 places.

$$\textit{The ratio is } \frac{10.5}{120} \textit{ which is 0.0875 .}$$

2. For the smaller triangle, write the ratio of the height to the base (bottom side). Calculate the decimal for this ratio to 4 places.

$$\textit{The ratio is } \frac{5.25}{60} \textit{ which is 0.0875 .}$$

In fact, all right triangles with a 5 degree angle are similar shapes and have the same value for this ratio (height to base).

3. Are these two triangles similar shapes.

Since they have the same ratio, yes they are similar shapes.

4. Use the ratio found above to find the height needed for a 16 foot ramp (192 inches).

$\frac{x}{192} = 0.0875$, *so* $x = 192(0.0875)$ *The height needed is 16.8 inches .*

5. Use the ratio found above to find the length (base) of a ramp that is 18 inches high.

$\frac{18}{x} = 0.0875$ *which means* $18 = 0.0875x$ *so* $x = \frac{18}{0.0875}$

The base needed is about 205.7 inches

6. The ratio of the height to the base is called the “tangent”. Your calculator has a button labeled “tan”. Use it to find the exact value of this ratio.

tan(5)= 0.087488664.

The basic keystrokes are TAN key 5 enter. You might need to put the mode for angle measures in degrees if your answer is different.

7. A triangle has a 30 degree angle. If the height is 3.2 meters, how long is the base?

Our calculator shows tan(30)= 0.57735.

So, $0.57735 = \frac{3.2}{x}$ *which means* $0.57735x = 3.2$ *Dividing gives 5.5425…*

The base is approximately 5.54 meters

We have said that the tangent is the ratio of height to base.
More specifically, the tangent is the ratio of the length of the opposite leg to the length of the adjacent leg.

$$\tan(x) = \frac{opposite\ leg}{adjacent\ leg}$$

Quick Check 1: A ramp for walkers is going to have a 10 degree angle. (A) Find the value of tan(10) to 5 places. (B) Use this ratio to find the height needed for a 60 foot ramp.

One measurement unit for an angle is degrees; often, this unit is shown by the symbol “°” sign; 5° means 5 degrees. This degree sign is like a small zero, but is not an exponent. We will use degrees to measure angles; in some situations “radians” are used to measure angles. The connection between the measures is that 360° is a full circle, and a full circle is 2π radians.

Using radian measure is very common in science. In later math classes, you will study “unit circles” as an approach to trigonometry … we are using a ‘right triangle’ approach. The basic trig functions are the same in both approaches.

Part B:

So far, we have used one ratio for sides in a right triangle. The hypotenuse (longest side) has not been involved with this ratio. We will look at two additional ratios where the hypotenuse is the denominator.

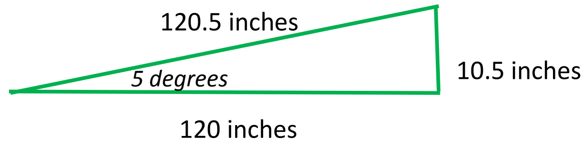
The sides of a right triangle can be compared in a total of 6 ways. Three of these are called “basic trigonometric functions”, with the definitions stated below:

$$\tan(x) = \frac{\textit{opposite leg}}{\textit{adjacent leg}}$$

$$\sin(x) = \frac{\textit{opposite leg}}{\textit{hypotenuse}} \qquad \cos(x) = \frac{\textit{opposite leg}}{\textit{hypotenuse}}$$

The remaining 3 ratios are the reciprocals of these 3 basic ratios, and are not used as frequently.

Here is the ramp situation we started with:



Questions:

8. Find the value of $\sin(5)$ to 4 decimal digits (nearest ten-thousandths)

sin(5) =0.087156, rounds to 0.0872

> We are starting to use the official definitions. In this case, the opposite leg is the height.

9. In the triangle, do the opposite leg and hypotenuse have the correct ratio?

This is the sine ratio, which is $\frac{10.5}{120.5}$ *and equals 0.08713...*

The ratio is very close (as close as we can get by measuring to the tenths).

10. Find the value of $\cos(5)$ to 4 decimal digits (nearest ten-thousandths)

cos(5) =0.996194698, rounds to 0.9962

> Now that we have checked the triangle with the trig functions, you can check the triangle with the Pythagorean Theorem:
> $a^2 + b^2 = c^2$

11. In the triangle, do the adjacent leg and hypotenuse have the correct ratio?

This is the cosine ratio, which is $\frac{120}{120.5}$ *and equals 0.99585...*

The ratio is very close (as close as we can get by measuring to the tenths).

12. An office building needs a wheelchair ramp with the 5 degree angle. If the base of the ramp is 16 feet, how long is the ramp itself?

We can use the cosine (adjacent leg to hypotenuse); cos(5) = 0.9962

$0.9962 = \frac{16}{x}$ *which means* $0.9962x = 16$ *and* $x = \frac{16}{0.9962}$ *=16.0610...*

The hypotenuse (ramp) is 16.06 feet

13. A triangle needs a 20° angle, using a 96 inch hypotenuse. What is the height of this triangle?

We can use the sine (opposite leg to hypotenuse); sin(20)=0.34202

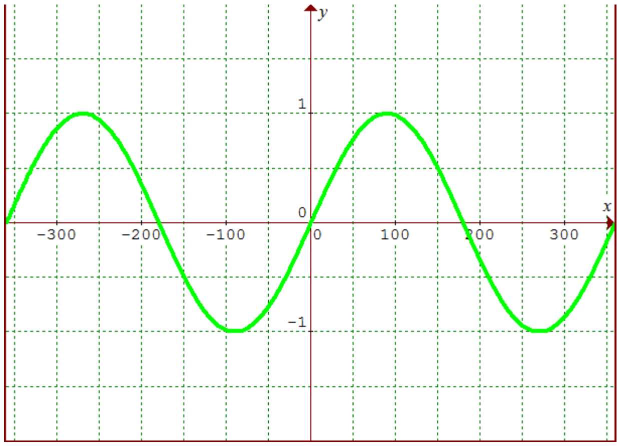
$0.34202 = \frac{x}{96}$*; so* $x = 0.34202(96)$ *=32.83392* *The height is 32.83 inches.*

Quick Check 2: A ladder 16 feet long is leaning against the side of a house. To be safe, the angle should be 75°. How far from the house should we place the bottom of the ladder?

Part C:

The basic trigonometric functions we have talked about are used as functions for graphing and other purposes. The graphs will use angles that would not make sense in a right triangle (90° or more). You might remember that a 180° forms a straight angle, a continuous straight line. Angles more than 180° can get close to being all the way around a circle … and 330° angle is almost a full circle. Angles more than 360° mean we go past a full circle; a 380° is the same as (360° + 20°)

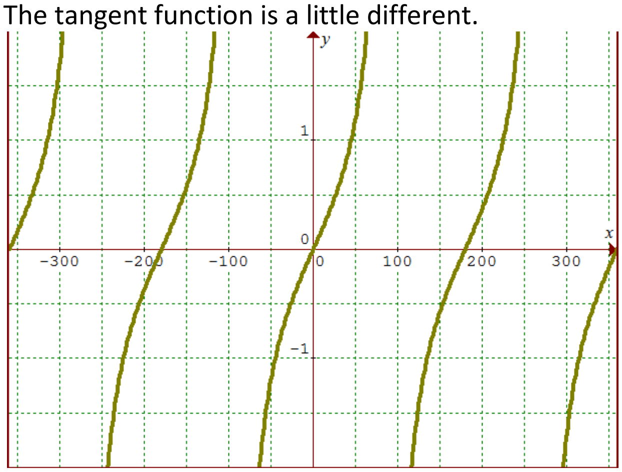
Here is a graph of $y = \sin(x)$ from -360 degrees to +360 degrees.



Trig functions produce a periodic graph, with repeating patterns. This sine function repeats every 360 degrees.

Trig functions have a period like this, where the graph repeats the pattern. The sine pattern is based on a circle: a group of 360 degrees, where the sine value is always between -1 and +1.

The tangent function is a little different.



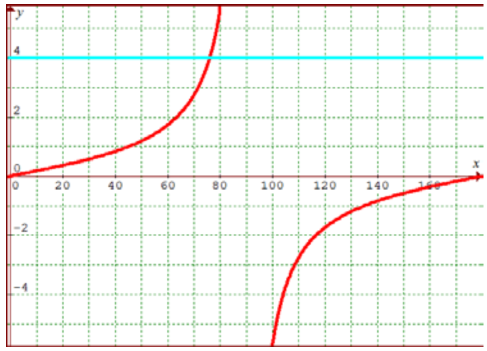
The tangent pattern is every 180 degrees. The value of the tangent ranges from negative infinity to positive infinity. We are going to use these graphs only in the domain of 0 degrees to 180 degrees, which will help us solve some problems.

Questions:
The safety instructions for ladders do not directly state a degree measurement. Instead, we are usually told that the ratio of height to base needs to be 4:1. We will find the angle required for a safe ladder.

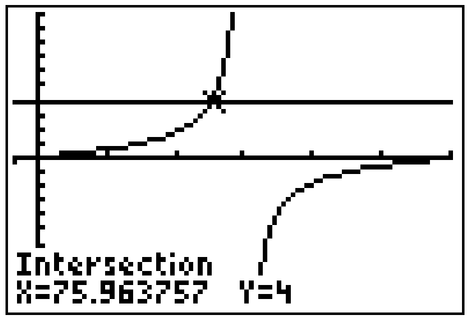
14. Use your calculator to graph $y = \tan(x)$ for the domain 0 to 180 degrees.

Our graph here is one piece of the repeating pattern. Most everyday situations involve angles between 0 and 180 degrees … though we will find many situations where larger angle sizes are needed.

15. The ratio of height to base needs to be 4:1, which means the required tangent value is 4.0. Enter $y = 4$ as a second function on your calculator.



16. The angle we need is represented by the intersection … where the tangent equals 4.0. Use the intersect program on your calculator to find that intersection.



The angle needs to be 75.9637… degrees; to the nearest degree … 76 degrees

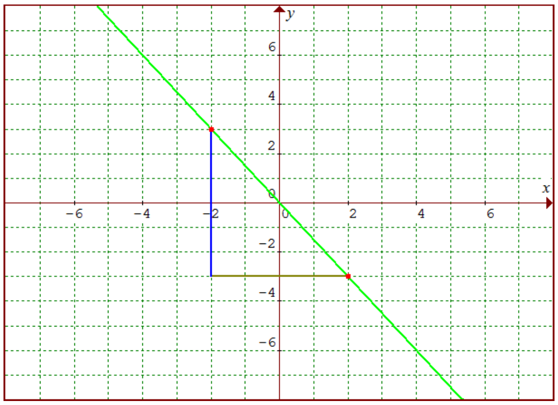
Quick Check 3: A family is taking a hike from the base of a mountain. The trail map says that the trail rises 800 vertical feet for each mile (5280 feet) horizontal distance. Treat these as the height and base of a right triangle. Use the intersect process to find the angle formed by the trail (to the nearest degree).

Part D:

You might have noticed that the phrase "height to base" is similar to another quantity we use for a linear function: Slope.
This is not a coincidence: Slope is a rate of change (y change over x change), and the tangent is a rate of change (vertical over horizontal). We will use this correspondence to explore the angle of a line.

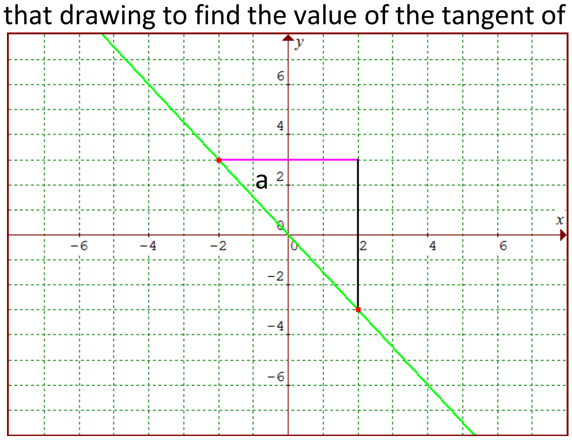
Review: $\tan(x) = \dfrac{\textit{opposite leg}}{\textit{adjacent leg}}$ $\quad\quad m = \dfrac{\Delta y}{\Delta x}$

17. This is the graph of a linear function. The graph also shows the base and height of a right triangle. Use the coordinates of those points to find the slope of the linear function.



The slope is $\frac{3-(-3)}{-2-2}=\frac{6}{-4}=-\frac{3}{2}$

18. This is the same function, with the base and height of a right triangle drawn. Use that drawing to find the value of the tangent of angle a.



The opposite leg is -6, the adjacent leg is +4; the tangent is $\frac{-6}{4}$ *=-1.5*

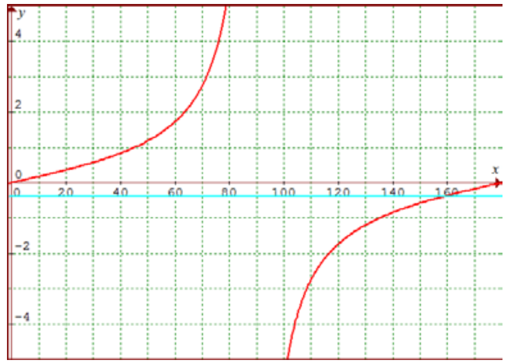
19. Is the tangent equal to the slope?

Yes, because $-\frac{3}{2}$ *= -1.5*

You might remember the relationship of the slopes for two perpendicular lines. The next problems explore that relationship by using the intersect process for tangent (slope) values.

20. The slope of line 1 is $-\frac{2}{5}$. Knowing that this is the tangent value, find the angle formed by this line.

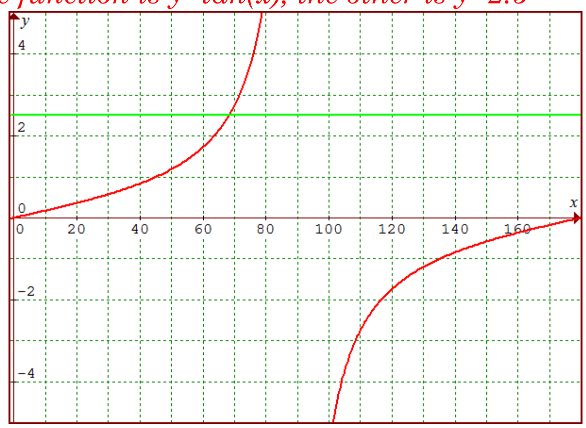
We need to find the angle so that tan(x) = -0.4
so one function is y=tan(x), the other is y=-0.4



Finding the intersection requires us to suggest the correct part of the domain.
Enter a number like 150 for the guess, or move the cursor closer.
The intersection occurs at about 158.198…
The angle is about 159 degrees

21. The slope of line 2 is $\frac{5}{2}$. Knowing that this is the tangent value, find the angle formed by this line.

One function is y=tan(x), the other is y=2.5



Finding the intersection requires us to suggest the correct part of the domain.
Enter a number like 60 for the guess, or move the cursor closer.
The intersection occurs at about 68.198… The angle is about 69 degrees

22. Find the difference in those angles. Are they perpendicular (meet at 90 degrees)?

The difference in angle measures is 159 – 69 =90 degrees
The lines are perpendicular.

We might remember that the slopes of perpendicular lines are negative reciprocals. Our work allows us to directly find the angle between the lines.

Quick Check 4: One line has a slope of 4. (A) Find the slope of a line perpendicular to this line. (B) Find the angles for the tangent, aka slope, of each line. (C) Does this confirm that the lines are perpendicular?