Lesson 3.x – Understanding Rational Exponents

Earlier, we used integer exponents for a number or variable base, like these:

4^3 $3x^{-2}$ $25n^3$

Positive exponents indicate a repeated product $25n^3 = 25 \cdot n \cdot n \cdot n$
Negative exponents indicate a division by a repeated product

$$3x^{-2} = 3 \cdot \frac{1}{x^2}$$

Separate from those situations, we saw a constant base with a variable exponent, like $y = 2^x$

Perhaps you wondered about the exponent values (x) that are fractional or decimal numbers. What does $2^{\frac{4}{5}}$ mean? How about $2^{1.6}$?

Part A:

As we often do, we will start with something we know: $4^3 = 64$

We also know that a whole number is equivalent to a fraction: $3 = \frac{3}{1}$

Putting both things together, we have $4^{\left(\frac{3}{1}\right)} = 64$

This expression includes parentheses around the fractional exponent $\left(\frac{3}{1}\right)$. Those parentheses are seldom written, so we need to remember that grouping.

The numerator in a fractional exponent indicates the POWER.

Let's explore what fractional exponents mean for the size of the result.

Questions:

1. Estimate the result for $4^{\frac{3}{2}}$

Since $\frac{3}{2}$ is between 1 and 2, the result must be between 4 and 16.

2. Is $9^{\frac{7}{4}}$ closer to 10 or is it closer to 70?

Since $\frac{7}{4}$ is closer to 2, $9^{\frac{7}{4}}$ is closer to 70 (note that $9^2 = 81$)

Quick Check 1: Estimate the value of $6^{\frac{9}{2}}$

Of course, we could just enter these expressions in a calculator or spreadsheet; that would provide a more precise 'answer', though we need to understand what fractional exponents mean.

Part B:

Think about pairs of opposite operations: add and subtract; multiply and divide. We are going to look at another pair of opposite operations: power and root.

We know that $4^{\left(\frac{3}{1}\right)} = 64$. What would $4^{\left(\frac{1}{3}\right)}$ mean?

Since $\frac{1}{3}$ is between 0 and 1, we know that $4^{\left(\frac{1}{3}\right)}$ is between 1 and 4.

How about $64^{\frac{1}{3}}$? We know the value is between 1 and 64; as an estimate, that is not very good at all.
The key thing is that opposite operations get us back to a starting value. Subtraction undoes an addition; dividing undoes a multiplying. What undoes a power?
A root undoes a power.

An exponent of $\frac{1}{3}$ must show the opposite operation of an exponent of 3.

The denominator in a fractional exponent indicates the ROOT.

Which root we are using is called the INDEX. The index is the denominator of the exponent.

This means that $64^{\frac{1}{3}} = 4$

Questions:

3. Find the value of $9^{\frac{3}{2}}$ without a calculator.

The denominator of the exponent is 2 (square root); the square root of 9 (=$\sqrt{9}$) is 3.
The numerator is 3 (cube), so $3^3 = 27$

You might try doing the power first (9 cubed) and then finding the square root. Do you get the same value? When there is only one base and one exponent, we can choose which order.

4. Write $4x^{\frac{1}{4}}$ as a radical

The denominator indicates the fourth root $4\sqrt[4]{x}$ *(only x has an exponent)*

5.Write $7\sqrt[3]{2y^4}$ in exponential form

The index is 3, and applies to $2y^4$ *, so we have* $7\left(2y^4\right)^{\frac{1}{3}}$

We'll consider later whether there are other ways to write the exponential expression.

Quick Check 2: (A) Find the value of $8^{\frac{2}{3}} \cdot 4^{\frac{3}{2}}$ without using a calculator.

(B) Write $12 + 5n^{\frac{3}{4}}$ using a radical

Part C:

One use of fractional exponents is with "Half-life" situations. For example, a particular medication might have a half-life of 4 hours; this means that each 4 hour period results in half of the medication remaining (and half is eliminated or used).

In general, half-life functions have the form $y = initial \cdot (0.5)^{\frac{time}{T\_Half}}$

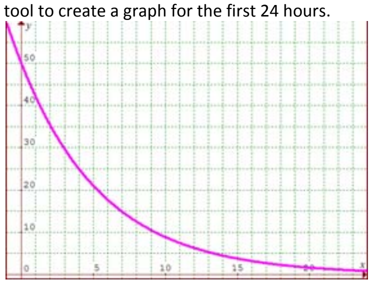
"Doubling time" works much the same way, where the function is

$y = initial\ value \bullet 2^{\frac{time}{T\_Dbl}}$

For the problems below, let's assume that there were 50 milligrams of the medication in the body at the start (hour 0). If the half life is 4 hours, this function gives the quantity remaining after *t* hours.

$$D(t) = 50(0.5)^{\frac{t}{4}}$$

Questions:
6. Use your graphing calculator or other tool to create a graph for the first 24 hours.



Half-life functions are a special case of exponential functions; half life involves exponential decay. We'll be learning more about exponential models … both growth and decay.

7. Calculate the medication remaining after 8 hours.

$D(8) = 50(0.5)^{\frac{8}{4}}$;

since 8/4 = 2, this is 50(0.5)²=50(.25)=12.5. Answer: 12.5 mg

8. Calculate the medication remaining after 12 hours.

$D(12) = 50(0.5)^{\frac{12}{4}}$;

since 12/4 = 3, this is 50(0.5)³=50(.125)=6.25. Answer: 6.25 mg

9. Check the half-life pattern: Was half of the medication eliminated from hour 8 to hour 12?

Since 6.25/12.5 = 0.5, yes … half of the medication was eliminated.

10. If the minimum effective level is 10 mg, how long is the 50 mg dose effective?

This can be estimated from the graph: 9 hours.

Quick Check 3: Iodine-131 has a half-life of about 8 days. If a sample starts with 200 grams, the function $R(t) = 200 \cdot 0.5^{\frac{t}{8}}$ shows the amount remaining after *t* days. Find the amount remaining after 16 days without using a calculator. Find the amount of iodine-131 remaining after 60 days using a calculator.

Part D:

On occasion, expressions involving rational exponents need to be simplified. The process involves two concepts … the domain of the expression, and simplifying the exponent.

$k^{\frac{12}{3}} = k^4$ for all values of k $\qquad$ $d^{\frac{10}{2}} = d^5$ for positive values of d

Note that $(-4)^{\frac{1}{2}}$ is not a real number … the answer is $2i$. In many areas of mathematics, the domain is restricted to values that produce real numbers. There are exceptions, though. More on this later.

Domain: $x^{\frac{1}{n}}$ results in a real number when n is odd OR when x is positive.

Simplifying: $x^{\frac{m}{n}}$ can be simplified when (1) n divides into m OR (2) when m and n have a common factor

Questions: For each expression, simplify if possible.

11. $8t^{\frac{6}{4}}$

The exponent has a common factor so this simplifies to $8t^{\frac{3}{2}}$
Note that 8 is not grouped with the base, and the domain is unchanged by simplifying.

12. $8 - x^{\frac{3}{9}}$

The exponent has a common factor so this simplifies to $8 - x^{\frac{1}{3}}$
The domain is unchanged by simplifying.

13. $10y^{\frac{2}{5}}$

The exponent does not have a common factor; the expression does not simplify.

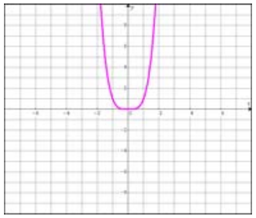
None of our problems have had grouping on the base; the process is a little different for $(9k)^{\frac{6}{4}}$. We'll look at those expressions a bit later in the course.

Quick Check 4: Simplify $d^{\frac{6}{3}} z^{\frac{2}{7}}$

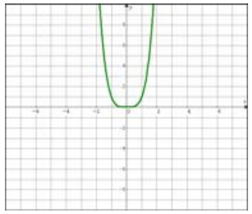
Part E:

Let us take a look at the graphs of some functions with fractional exponents … notice the domain, the shapes, and the effect of adding a constant in the base.

We said that $k^{\frac{12}{3}} = k^4$ for all values of k. Here are the graphs of $y = x^{\frac{12}{3}}$ and $y = x^4$



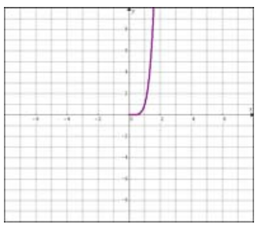
$y = x^{\frac{12}{3}}$



$y = x^4$

The index of three (cube root) means that the domain is all real numbers, which results in the graphs being the same.

We said that $d^{\frac{10}{2}} = d^5$ for positive values of d. Here are the graphs of each expression.



$y = x^{\frac{10}{2}}$

$y = x^5$

The green portion of the second graph is the 'new' part … the domain includes negative x values for $y = x^5$; the domain is "0 or greater" for the square root function shown.

The convention is that function values need to be real numbers. For 'even' roots, this means that the base can not be negative.

Questions: For each function, identify the domain.

14. $H(n) = (n+1)^{\frac{3}{2}}$

The index is 2; this is the square root of (n + 1), which is then cubed.
The base will be 0 if n is -1, and positive for values greater than -1.
The domain is $n \geq -1$*; in interval notation, this is [-1, ∞)*

15. $P(x) = \sqrt[4]{x-2}$

The index is 4; this is the fourth root of (x – 2).
The base will be 0 if n is 2, and positive for values greater than 2.
The domain is $x \geq 2$*; in interval notation, this is [2, ∞)*

16. $g(k) = (k+3)^{\frac{2}{3}}$

The index is 3; this is the cube root of (k + 3), which is then squared.
All cube roots are real numbers for a real number input.
The domain is all real numbers; in interval notation, this is (-∞, ∞)

Take a minute to graph the function for #14 and #15 on your calculator or software. We will use the idea of the "first point" on the graph of even root radical functions … the point where the function starts or stops.

Questions: For each function, identify the coordinates of the 'starting point' on the graph.

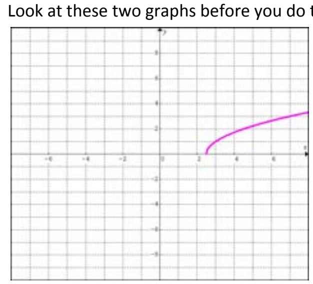
17. $f(x) = \sqrt{x+3}$

The index is 2; this is the square root of (x + 3).
The base will be 0 if x is -3; if x =-3, then the function value is 0.
The coordinates of the starting point are (-3, 0)

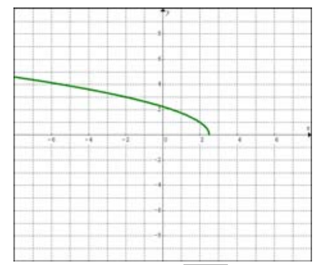
17. $h(t) = \sqrt{2t-5}$

The index is 2; this is the square root of (2t – 5).
The base will be 0 if t is $\frac{5}{2}$*; if t =2.5, then the function value is 0.*
The coordinates of the starting point are (2.5, 0)

Look at these two graphs before you do the Quick Check.



$h(t) = \sqrt{2t-5}$



$k(t) = \sqrt{5-2t}$

(2.5, 0) is the 'starting point' for both graphs. The domain of *h* is [2.5, ∞); the domain of *k* is (-∞, 2.5] .

The formal statement is that there is a horizontal shift of 2.5 for both functions.

Notice the different 'direction' caused by the coefficient of the variable inside the radical.

Quick Check 5: For the function $R(d) = (7-2d)^{\frac{1}{2}}$

(A) State the domain; write the domain in interval notation.
(B) Identify the coordinates of the 'starting point'.